Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number	Candidate Number

Sample Assessment Material

(Time: 1 hour 30 minutes) Paper Reference **9FM0/4A**

Further Mathematics

Advanced

Paper 4A: Further Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B)
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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Pearson

Answer ALL questions. Write your answers in the spaces provided.

1. (a) Show that the transformation

$$w = \frac{z-1}{z} \qquad z \neq 0$$

maps the locus $|z - 1| = 1$ in the z-plane onto the locus $|w| = |w - 1|$ in the w-plane.

(3)

The region $|z - 1| \leqslant 1$ in the z-plane is mapped onto the region T in the w-plane.

(b) Shade the region T on an Argand diagram.

(2)

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1a. $w = \frac{z-1}{z}$

$wz = z - 1$ rearrange for z

$z - zw = 1$

$z(1-w) = 1$

$z = \frac{1}{1-w}$

$|z-1| = 1$

$\left|\frac{1}{1-w} - 1\right| = 1$

$\left|\frac{1-(1-w)}{1-w}\right| = 1$

$\left|\frac{1-1+w}{1-w}\right| = 1$

$\left|\frac{w}{1-w}\right| = 1$ split using rule $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$\frac{|w|}{|1-w|} = 1$

$|w| = |1-w|$

$|w| = |(-1)(w-1)|$ split using rule $|ab| = |a||b|$

$|w| = |-1||w-1|$

$|w| = 1|w-1|$

$|w| = |w-1|$

$|w| = |w-1|$

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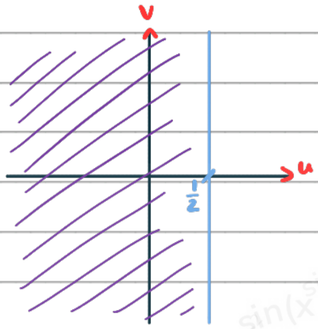
b. $|w| \leq |w-1|$

perpendicular bisector between $(0,0)$ and $(1,0)$

shade in region containing $(0,0)$ since lies on ... $\leq$



→ plot $(0,0)$ and $(1,0)$

→ find midpoint of both these points

$$\left(\frac{0+1}{2}, \frac{0+0}{2}\right) = \left(\frac{1}{2}, 0\right)$$

→ find gradient of line containing $(0,0)$ and $(1,0)$

$$M = \frac{0-0}{1-0} = \frac{0}{1} = 0$$

→ find gradient of perpendicular bisector.

since it is perpendicular to line containing $(0,0)$ and $(1,0)$, :

$$M_{tangent} \times M_{normal} = -1$$

$$0 \times M_{normal} = -1$$

$$M_{normal} = \frac{-1}{0} = \text{undefined}$$

↳ will be a vertical line crossing through midpoint

2.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}$$

(a) Verify that $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and find the corresponding eigenvalue.

(3)

(b) Show that 9 is another eigenvalue of $\mathbf{A}$ and find a corresponding eigenvector.

(5)

Given that a third eigenvector of $\mathbf{A}$ is $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

(c) write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$$

(4)

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2a. $A\mathbf{x} = \lambda \mathbf{x}$

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

sub in eigenvectors and solve for λ

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} (1)(2)+(0)(-2)+(4)(1) \\ (0)(2)+(5)(-2)+(4)(1) \\ (4)(2)+(4)(-2)+(3)(1) \end{pmatrix} = \begin{pmatrix} 2\lambda \\ -2\lambda \\ \lambda \end{pmatrix}$$

$$\begin{pmatrix} 6 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ -2\lambda \\ \lambda \end{pmatrix}$$

$6 = 2\lambda$

$\lambda = 3$

3 is corresponding eigenvalue //

b. characteristic eq^n: $\det(A - \lambda I) = 0$

if 9 is an eigenvalue $\det(A - 9I) = 0$

$$A - \lambda I = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix} - 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix} - \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

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$$A - \lambda I = \begin{pmatrix} 1-9 & 0 & 4 \\ 0 & 5-9 & 4 \\ 4 & 4 & 3-9 \end{pmatrix} = \begin{pmatrix} -8 & 0 & 4 \\ 0 & -4 & 4 \\ 4 & 4 & -6 \end{pmatrix}$$

$$\det\left[\begin{pmatrix} -8 & 0 & 4 \\ 0 & -4 & 4 \\ 4 & 4 & -6 \end{pmatrix}\right] = 0$$

$$(-8)\begin{vmatrix} -4 & 4 \\ 4 & -6 \end{vmatrix} \cancel{- (0)\begin{vmatrix} 0 & 4 \\ 4 & -6 \end{vmatrix}} + (4)\begin{vmatrix} 0 & -4 \\ 4 & 4 \end{vmatrix} = 0$$

$$(-8)\left[(-4)(-6) - (4)(4)\right] + (4)\left[\cancel{(0)(4)} - (-4)(4)\right] = 0$$

$$(-8)(8) + (4)(16) = 0$$

$\therefore 9$ is an eigenvalue

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x + 0y + 4z = 9x$$
$$0x + 5y + 4z = 9y$$
$$4x + 4y + 3z = 9z$$

$$-8x + 4z = 0 \quad (1)$$
$$-4y + 4z = 0 \quad (2)$$
$$4x + 4y - 6z = 0 \quad (3)$$

(1) $-8x + 4z = 0$

$8x = 4z$

$2x = z \Rightarrow x = \frac{1}{2}z$

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(2) $-4y+4z=0$

$4y=4z$

$y=z$

can form a general eigenvector eqⁿ in terms of z: $\begin{pmatrix} \frac{1}{2}z \\ z \\ z \end{pmatrix}$ z can be any no.

e.g. let $z=2$, corresponding eigenvector: $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ //

c. firstly find eigenvalue corresponding to eigenvector $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

↳ use same method as part (a)

$Ax=\lambda x$

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

sub in eigenvectors and solve for λ

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}=\lambda\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} (1)(2)+(0)(1)+(4)(-2) \\ (0)(2)+(5)(1)+(4)(-2) \\ (4)(2)+(4)(1)+(3)(-2) \end{pmatrix}=\begin{pmatrix} 2\lambda \\ \lambda \\ -2\lambda \end{pmatrix}$$

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$$\begin{pmatrix} -6 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ \lambda \\ -2\lambda \end{pmatrix}$$

$\lambda = -3$ ← final eigenvalue

$$\boxed{D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad P = \begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix}}$$

D, eigenvalues can be in any order as long as diagonal

P, eigenvectors in column must match corresponding eigenvalues.

3. (i) Lottery X requires each player to buy a ticket and choose 5 different numbers from the numbers 1 to 45 inclusive.
Lottery Y requires each player to buy a ticket and choose 6 different numbers from the numbers 1 to 35 inclusive.
A player wins if their chosen numbers match completely those drawn at random by the lotteries.
A person wishes to play one of these two lotteries.

The price of a ticket to play each lottery is the same.
The prize money for winning each lottery is the same.

Decide which lottery you would recommend that they play, giving a reason for your answer.

(2)

(ii) Use Fermat's little theorem to show that when 128^{129} is divided by 17 the remainder is 9

(4)

(iii) There are $3x$ chairs in a room. When these chairs are set out in rows of 7 there are two chairs left over.

(a) Form and solve a congruence equation for x

(3)

Given that there are at least 100 chairs and that one third of the chairs can be arranged exactly into 5 equal rows,

(b) find the least possible number of chairs in the room.

(3)

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3i. Lottery X: choose 5 no.s from 45 $\binom{45}{5} = 1{,}221{,}759$

Lottery Y: choose 6 no.s from 35 $\binom{35}{6} = 1{,}623{,}160$

$\therefore$ probability of winning X $= 1/1{,}221{,}759$

$\therefore$ probability of winning Y $= 1/1623160$

$1/1{,}221{,}759 > 1/1623160$

$\therefore$ Lottery X has better odds

ii. Fermat's Little Theorem: $a^{16} \equiv 1 \pmod{17}$

$128^{129} = (128^{16})^8 \times 128$

$(128)^{129} \pmod{17}$

$(128^{16})^8 \times 128 \pmod{17}$

$1 \times 128 \pmod{17}$

$128 \pmod{17} \equiv 9$

$\therefore$ remainder is 9

iiia. congruence eq^n: $3x \equiv 2 \pmod 7$

gcd (3,7)

$7 = 2(3) + 1 \rightarrow 1 = 7 - 2(3)$ $\quad$ $3(-2) + 7(1) = 1$

$3 = 3(1)$ $\quad$ $x = -2,\ y = 1$

$3x + 7y = 1$

$3x = 1 \pmod 7$

$3(-2) = 1 \pmod 7$

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-2 is multiplicative inverse

$(-2) \times 3x \equiv (-2) \times 2 \pmod 7$

$x \equiv -4 \pmod 7$

$x \equiv 3 \pmod 7$ //

b. $3x \geq 100$ $\quad x \geq \frac{100}{3}\ (33.\dot{3})$ $\quad x \geq 34$

$\cancel{\frac{1}{3}}(3x)$ can be arranged in 5 equal rows (divisible by 5)

└ $x \equiv 0 \pmod 5$

$x \equiv 3 \pmod 7$

└ must find values of x here also divisible by 5.

$x = 3, 10, 17, 24, 31, 38, \underline{\underline{45}}$

$3x = 3(45) = 135$

Least no. of chairs in room $= 135$ //

4. (i) Two distinct elements of a group G are a and b.
The element a has order 5 and $a^3b = ba^3$
Prove that $ab = ba$

(4)

Given that p, q, r and s are distinct elements

(ii) (a) check each of the group axioms for the set $A = \{p, q, r, s\}$ under the operation $\oplus$ defined in the table below.

(b) Hence determine whether the set A forms a group under the operation $\oplus$.

$\oplus$	$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{s}$
$\boldsymbol{p}$	p	q	r	s
$\boldsymbol{q}$	q	p	q	r
$\boldsymbol{r}$	r	q	p	q
$\boldsymbol{s}$	s	r	q	p

(3)

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4i. a has order 5: $a^5 = e$ also means $a^6 = a$

$$a^3 b = ba^3$$
$$(a^3 b)a^3 = (ba^3)a^3$$
$$a^3 ba^3 = ba^6$$
$$a^6 b = ba^6$$
$$ab = ba$$

iia. To prove if A is a group, must check the following:

→ closure: all transformations in Cayley table are in set A, so closed ✓

→ identity: P is the identity ✓

→ inverse: P is its own self inverse
q is its own self inverse
r is its own self inverse
s is its own self inverse ✓

→ associativity: $(q \oplus r) \oplus s = q \oplus s = r$

$q \oplus (r \oplus s) = q \oplus q = P$

$(q \oplus r) \oplus s \neq q \oplus (r \oplus s)$ ∴ NOT ASSOCIATIVE X

b. since A is NOT associative, set A is NOT a group

5.

$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx, \qquad n \geqslant 0$$

(a) Prove that, for $n \geqslant 2$

$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

(5)

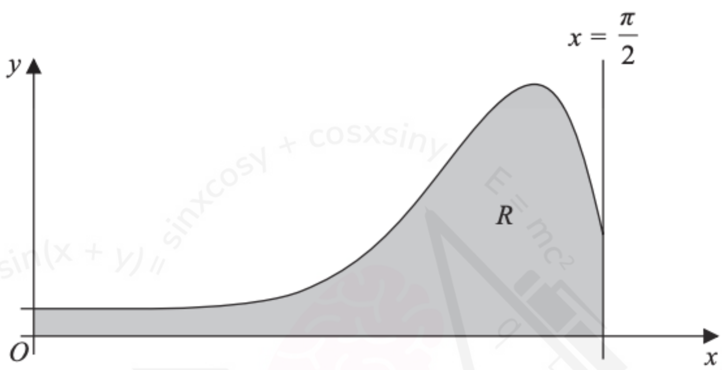

Figure 1

Figure 1 shows the vertical cross-section of a proposed flood defence system. The cross-section of the flood defence system is modelled by the curve with equation

$$y = 1.2x^6 \cos x + 0.2 \qquad 0 \leqslant x \leqslant \frac{\pi}{2}$$

where x and y are measured in metres.

The area R, shown shaded in Figure 1, is bounded by the curve, the y-axis, the x-axis and the line with equation $x = \frac{\pi}{2}$

The flood defence system will come in hollow sections that will be filled with water once they are in place. Each section will have a length of 10 metres.

(b) Use the model and the answer to part (a), to estimate the volume of water needed to fill each section.

(6)

Each section can be filled with water at a maximum rate of 175 litres per minute and is required to be filled with water within 1 hour of being put in place.

(c) Use the model to decide whether this requirement can be met, showing all your reasoning.

(2)

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6.1: Reduction Formulae

5a. $I_n = \int_0^{\frac{\pi}{2}} \underbrace{x^n}_{u} \underbrace{\cos x}_{v'} \, dx$

pick $u = x^n$ because we don't know how to integrate if we picked $v' = x^n$

Integration by parts:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Given in formulae booklet

$u = x^n \qquad v' = \cos x$

$u' = nx^{n-1} \qquad v = \sin x$

$$I_n = x^n \sin x - \int_0^{\frac{\pi}{2}} nx^{n-1} \sin x \, dx$$

$$I_n = x^n \sin x - n \int_0^{\frac{\pi}{2}} \underbrace{x^{n-1} \sin x \, dx}_{\text{integrate by parts again}}$$

$$\int \underbrace{x^{n-1}}_{u} \underbrace{\sin x}_{v'} \, dx$$

$u = x^{n-1} \qquad v' = \sin x$

$u' = (n-1)x^{n-2} \qquad v = -\cos x$

$$\int x^{n-1} \sin x \, dx = -x^{n-1} \cos x - \int -\cos x \, (n-1) x^{n-2} \, dx$$

take $-(n-1)$ out of integral

$$-x^{n-1} \cos x + (n-1) \int x^{n-2} \cos x \, dx$$

$$I_n = x^n \sin x - n \left[-x^{n-1} \cos x + (n-1) \int x^{n-2} \cos x \, dx \right]$$

$$I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x \, dx$$

apply limits from integral

$$I_n = \left[x^n \sin x + nx^{n-1} \cos x \right]_0^{\frac{\pi}{2}} - n(n-1) I_{n-2}$$

$$I_n = \left[\left(\left(\tfrac{\pi}{2}\right)^n \sin\left(\tfrac{\pi}{2}\right) + n\left(\tfrac{\pi}{2}\right)^{n-1} \cos\left(\tfrac{\pi}{2}\right) \right) - \left((0)^n \sin(0) + n(0)^{n-1} \cos(0) \right) \right] - n(n-1) I_{n-2}$$

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$$I_n = \left[\left(\left(\tfrac{\pi}{2}\right)^n(1) + \cancel{n\left(\tfrac{\pi}{2}\right)^{n-1}(0)}\right) - \left(\cancel{(0)^n(0) + n(0)^{n-1}}\right)\right] - n(n-1)I_{n-2}$$

$$I_n = \left(\tfrac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

b. use limits $x=0$ and $x=\frac{\pi}{2}$

$$A = \int_0^{\frac{\pi}{2}} 1.2x^6\cos(x) + 0.2 \; dx$$

$$= \int_0^{\frac{\pi}{2}} 1.2x^6\cos(x)\, dx + \int_0^{\frac{\pi}{2}} 0.2 \, dx$$

$$= 1.2\int_0^{\frac{\pi}{2}} x^6\cos(x)\, dx + \left[0.2x\right]_0^{\frac{\pi}{2}}$$

$$A = 1.2\, I_6 + \left[\left(0.2 \times \tfrac{\pi}{2}\right) - \cancel{(0.2 \times 0)}\right]$$

$$A = 1.2\, I_6 + \frac{\pi}{10}$$

$$I_6 = \left(\tfrac{\pi}{2}\right)^6 - 6(6-1)I_4 = \frac{\pi^6}{64} - 30\, I_4$$

$$I_4 = \left(\tfrac{\pi}{2}\right)^4 - 4(4-1)I_2 = \frac{\pi^4}{16} - 12\, I_2$$

$$I_2 = \left(\tfrac{\pi}{2}\right)^2 - 2(2-1)I_0 = \frac{\pi^2}{4} - 2\, I_0$$

Work out I_0 directly:

$$I_0 = \int_0^{\frac{\pi}{2}} x^0\cos x\, dx = \int_0^{\frac{\pi}{2}} \cos x \, dx = \left[\sin x\right]_0^{\frac{\pi}{2}}$$

$$= \left[\left\{\sin\left(\tfrac{\pi}{2}\right)\right\} - \left\{\sin(0)\right\}\right]$$

$$= 1$$

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$I_2 = \frac{\pi^2}{4} - 2(1) = \frac{\pi^2}{4} - 2$

$I_4 = \frac{\pi^4}{16} - 12\left(\frac{\pi^2}{4} - 2\right) = \frac{\pi^4}{16} - 3\pi^2 + 24$

$I_6 = \frac{\pi^6}{64} - 30\left(\frac{\pi^4}{16} - 3\pi^2 + 24\right) = \frac{\pi^6}{64} - \frac{15\pi^4}{8} + 90\pi^2 - 720$

$A = 1.2\left(\frac{\pi^6}{64} - \frac{15\pi^4}{8} + 90\pi^2 - 720\right) + \frac{\pi}{10}$

$A = 1.087027136 m^2$

must do cross-sectional area × length = volume

10m ← Given in Q

Volume $= 1.087027136 \times 10$

$= 10.87027136 m^3$

$10.9 m^3$ (3s.f.)

c. $10.9 m^3 = 10900$ litres

$\frac{10900 \text{ l}}{175 \text{ l min}^{-1}} = \frac{436}{7}$ min ≈ 62.3 min (3s.f.)

$62.3 > 60$

∴ will take over an hour to fill

∴ requirement cannot be met

6.

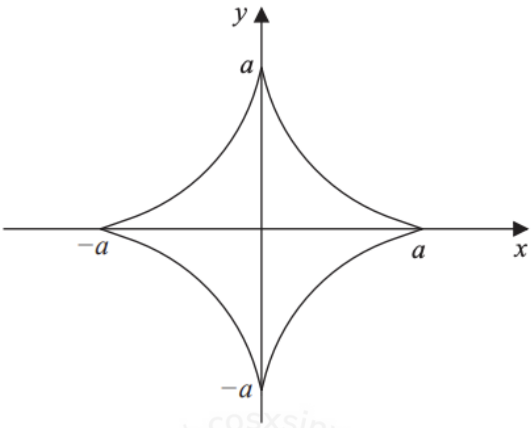

Figure 2

Figure 2 shows the curve with parametric equations

$$x = a\cos^3\theta \qquad y = a\sin^3\theta \qquad\qquad 0 \leqslant \theta < 2\pi$$

where $a > 0$

(a) Find the total length of this curve in terms of a.

(7)

The curve is used to model the design for a new sweet. The curve is rotated through π radians about the x-axis to create the shape of a sweet. The sweet is to be covered in chocolate.

Given that the total length of the curve is 5 cm,

(b) estimate the surface area of the sweet that is to be covered in chocolate.

(6)

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6 a.

$$S = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$ GIVEN IN FORMULAE BOOKLET

$x = a\cos^3\theta$

$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$

$\left(\frac{dx}{d\theta}\right)^2 = (-3a\cos^2\theta\sin\theta)^2 = 9a^2\cos^4\theta\sin^2\theta$

$y = a\sin^3\theta$

$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$\left(\frac{dy}{d\theta}\right)^2 = (3a\sin^2\theta\cos\theta)^2 = 9a^2\sin^4\theta\cos^2\theta$

$$S = \int \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\, d\theta$$

$$S = \int \sqrt{9a^2\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)}\, d\theta$$

remember $\cos^2\theta + \sin^2\theta = 1$

$$S = \int \sqrt{9a^2\cos^2\theta\sin^2\theta(1)}\, d\theta$$

↓ split surd

$$S = \int \sqrt{9}\sqrt{a^2}\sqrt{\cos^2\theta}\sqrt{\sin^2\theta}\, d\theta$$

$$S = \int 3a\cos\theta\sin\theta\, d\theta$$

$$S = 3a\int \cos\theta\sin\theta\, d\theta$$

$$S = 3a\left[\tfrac{1}{2}\sin^2\theta\right]$$

We need to find length of curve

↳ we can find length from $(0,a)$ to $(a,0)$ // then $\times 4$ to find entire length

must find $(0,a)$ and $(a,0)$ in terms of θ (parametric)

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$x = 0$

$0 = a\cos^3\theta$

$0 = \cos^3\theta$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}$

$x = a$

$a = a\cos^3\theta$

$\cos^3\theta = 1$

$\cos\theta = \sqrt[3]{1} = 1$

$\theta = 0$

$S = 4 \times 3a\left[\frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}}$

$S = 12a\left[\left(\frac{1}{2}\right) - (0)\right]$

$S = 12a(1/2)$

$S = 6a$

length of curve $= 6a$ //

b. Q states length $= 5$

$\therefore 6a = 5$

$a = 5/6$

$S_x = 2\pi \int y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ FORMULAE BOOKLET

use same limits as part (a)

use working from part (a) but now sub in $a = 5/6$

$\left(\frac{dx}{d\theta}\right)^2 = 9(5/6)^2\cos^4\theta\sin^2\theta = \frac{25}{4}\cos^4\theta\sin^2\theta$

$\left(\frac{dy}{d\theta}\right)^2 = 9(5/6)^2\sin^4\theta\cos^2\theta = \frac{25}{4}\sin^4\theta\cos^2\theta$

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$$S_x = 2\pi\int_0^{\frac{\pi}{2}} \left(\tfrac{5}{6}\right)\sin^3\theta\sqrt{\tfrac{25}{4}\cos^4\theta\sin^2\theta + \tfrac{25}{4}\sin^4\theta\cos^2\theta}\,d\theta$$

$$S_x = 2\pi\int_0^{\frac{\pi}{2}} \left(\tfrac{5}{6}\right)\sin^3\theta\sqrt{\tfrac{25}{4}\cos^2\theta\sin^2\theta\left(\cos^2\theta+\sin^2\theta\right)}\,d\theta$$

$$S_x = 2\pi\int_0^{\frac{\pi}{2}} \left(\tfrac{5}{6}\right)\sin^3\theta\sqrt{\tfrac{25}{4}}\sqrt{\cos^2\theta}\sqrt{\sin^2\theta}\,d\theta$$

$$S_x = 2\pi\int_0^{\frac{\pi}{2}} \left(\tfrac{5}{6}\right)\sin^3\theta\left(\tfrac{5}{2}\right)\cos\theta\sin\theta\,d\theta$$

factor out $(5/6)(5/2)$

$$S_x = \frac{25\pi}{6}\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\,d\theta$$

$$S_x = \frac{25\pi}{6}\left[\frac{1}{5}\sin^5\theta\right]_0^{\frac{\pi}{2}}$$

$$S_x = \frac{25\pi}{6}\left[\frac{1}{5}\sin^5\left(\frac{\pi}{2}\right) - \frac{1}{5}\sin^5(0)\right]$$

$$S_x = \frac{25\pi}{6}\left(\frac{1}{5}\right)$$

$$S_x = \frac{5\pi}{6}$$

$$\frac{5\pi}{6} \times 2 = \frac{5\pi}{3}$$

multiply by 2 to account for other quadrant rotated

Surface area ≈ $5.24\,cm^2$ (3 s.f.)

7. (i) Solve the recurrence relation

$$U_{n+2} = 6U_{n+1} - 9U_n + 4 \qquad n \geqslant 1$$

$$U_1 = 4 \qquad U_2 = 7$$

(7)

(ii) A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$(n+1)x_{n+2} + x_{n+1} - (n+2)x_n = 0 \qquad n \geqslant 1$$

$$x_1 = 2 \qquad x_2 = 5$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$x_n = \frac{1}{8}[25 + 3(1 + 2n)(-1)^n]$$

(6)

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7i. Associated homogenous recurrence relation:

$u_{n+2} = 6u_{n+1} - 9u_n$

$u_{n+2} - 6u_{n+1} + 9u_n = 0$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3$

Complementary function (C.F.):

$u_n = (A+Bn)(3^n)$

Try the particular integral (P.I.): $u_n = \lambda$

$u_{n+1} = \lambda$

$u_{n+2} = \lambda$

$\lambda = 6\lambda - 9\lambda + 4$

$4\lambda = 4$

$\lambda = 1$

P.I. $= 1$

gen sol^n: C.F. + P.I

$u_n = (A+Bn)(3^n) + 1$

Using boundary conditions, to form 2 eq^{n}s containing A and B and solve simultaneously

$u_1 = 4$

$u_1 = (A + B(1))(3^1) + 1 = 4$

$4 = (A+B)3 + 1$

$3A + 3B = 3$ ①

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$u_2 = 7$

$u_2 = (A + B(2))(3^2) + 1 = 7$

$7 = (A + 2B)9 + 1$

$9A + 18B = 6$

$3A + 6B = 2$ ②

$3A + 3B = 3$

$3A + 6B = 2$ ⊖

$-3B = 1$

$B = -1/3$

$3A + 3B = 3$

$3A + 3(-1/3) = 3$

$3A - 1 = 3$

$3A = 4$

$A = 4/3$

$A = 4/3$ and $B = -1/3$

$u_n = \left(\frac{4}{3} - \frac{1}{3}n\right)(3^n) + 1$

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ii. Basis step:

let $n=1$

$$X_1 = \frac{1}{8}\left[25 + 3(1+2(1))(-1)^1\right]$$
$$X_1 = \frac{1}{8}\left[25 + 3(3)(-1)\right]$$
$$X_1 = \frac{1}{8}\left[25-9\right]$$
$$X_1 = \frac{1}{8}\left[16\right]$$
$$X_1 = 2$$

let $n=2$

$$X_2 = \frac{1}{8}\left[25 + 3(1+2(2))(-1)^2\right]$$
$$X_2 = \frac{1}{8}\left[25 + 3(5)(1)\right]$$
$$X_2 = \frac{1}{8}\left[25+15\right]$$
$$X_2 = \frac{1}{8}\left[40\right]$$
$$X_2 = 5$$

$\therefore$ true for $n=1$ and $n=2$

Assumption step:

assume $n=k$, $X_k = \frac{1}{8}\left[25 + 3(1+2k)(-1)^k\right]$

$n=k+1$, $X_{k+1} = \frac{1}{8}\left[25 + 3(1+2\{k+1\})(-1)^{k+1}\right]$

Inductive step:

$$(k+1)X_{k+2} + X_{k+1} - (k+2)X_k = 0$$
$$(k+1)X_{k+2} + \frac{1}{8}\left[25 + 3(1+2\{k+1\})(-1)^{k+1}\right] - (k+2)\frac{1}{8}\left[25 + 3(1+2k)(-1)^k\right] = 0$$
$$(k+1)X_{k+2} = (k+2)\frac{1}{8}\left[25 + 3(1+2k)(-1)^k\right] - \frac{1}{8}\left[25 + 3(1+2\{k+1\})(-1)^{k+1}\right]$$
$$(k+1)X_{k+2} = \frac{1}{8}\left\langle \underbrace{(k+2)\left[25 + 3(1+2k)(-1)^k\right]}_{\text{expand not fully}} - \left[25 + 3(2k+3)(-1)^{k+1}\right] \right\rangle$$
$$(k+1)X_{k+2} = \frac{1}{8}\left\langle 25(k+2) + 3(k+2)(1+2k)(-1)^k - 25 - 3(2k+3)(-1)^{k+1} \right\rangle$$

simplify pink

$$(k+1)X_{k+2} = \frac{1}{8}\left\langle 25(k+1) + 3(k+2)(1+2k)(-1)^k - 3(2k+3)(-1)^{k+1} \right\rangle$$

factor out $3(-1)^{k+2}$ — we are trying to work towards $k+2$

$$(k+1)X_{k+2} = \frac{1}{8}\left\langle 25(k+1) + 3(-1)^{k+2}\left[(k+2)(1+2k)(-1)^{-2} - (2k+3)(-1)^{-1}\right] \right\rangle$$

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$$(k+1)x_{k+2} = \frac{1}{8}\left\langle 25(k+1) + 3(-1)^{k+2}\left[(k+2)(1+2k)(1) - (2k+3)(-1)\right]\right\rangle$$

simplify

$$(k+1)x_{k+2} = \frac{1}{8}\left\langle 25(k+1) + 3(-1)^{k+2}\left[2k^2+5k+2+2k+3\right]\right\rangle$$

factorise

$$(k+1)x_{k+2} = \frac{1}{8}\left\langle 25(k+1) + 3(-1)^{k+2}\left[2k^2+7k+5\right]\right\rangle$$

$$\cancel{(k+1)}x_{k+2} = \frac{1}{8}\left\langle 25\cancel{(k+1)} + 3(-1)^{k+2}(2k+5)\cancel{(k+1)}\right\rangle$$

divide everything by $(k+1)$

$$x_{k+2} = \frac{1}{8}\left\langle 25 + 3(2k+5)(-1)^{k+2}\right\rangle$$

$$x_{k+2} = \frac{1}{8}\left[25 + 3(1+2(k+2))(-1)^{k+2}\right] \;//$$

Result holds for $n=k+2$

Conclusion:

If the formulae is true for $n=k$ and $n=k+1$, then it is shown to be true for $n=k+2$. As the result is true for $n=1$ and $n=2$ it is now also true for all $n \in \mathbb{Z}^+$ by mathematical induction //